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# LEARNING MATHEMATICS FOR TEACHING: FROM A TEACHERS' SEMINAR TO THE CLASSROOM

ABSTRACT. A successful practice grounded in the principles that guide the current mathematics education reform effort requires a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess. However, it is not as clear how teachers' mathematical understandings develop and how those understandings affect instruction. This paper explores two avenues for K-6 teachers' mathematical development, (a) engagement in inquiry into mathematics itself, and (b) investigation of children's mathematical thinking, illustrating how the need for these two kinds of investigations arises in classroom situations and how they can be pursued in a professional development setting.

## INTRODUCTION

The year Anne Marie O'Reilly began teaching sixth grade, she was working to develop a practice that built from her students' ideas. She had studied reform documents, had read Dewey and Piaget, among others, and had thought hard about the implications for her classroom of a constructivist view of learning. But, with all her preparation – and having taken several workshops and courses about learning and teaching mathematics – that first year in teaching sixth grade was difficult and unsatisfying. O'Reilly illustrated her struggles in this vignette, drawn from a lesson on fractions:

> The class had no problem telling me how to write the fraction for one-half on the board – "1/2." Then I asked my students what the top and bottom numbers in that fraction stood for. I was not prepared for David's answer:
>
> 1 thing
> – cut into
> 2 equal pieces
>
> David made it clear, as he drew the horizontal bar in the air with his finger, that the line took the place of the words "cut into."
>
> I had done my homework. I had consulted what I considered to be a reliable resource for helping me to understand the mathematics involved in what I was teaching. . . . What I wanted to hear was something to the effect that the top number tells how many things we have and the bottom number tells what fractional part is being counted. I was stunned and at a loss as to how I should challenge David to help him say what I wanted to hear. I

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looked to his classmates to offer a challenge and asked if anyone else wanted to give an explanation. . . . To my dismay, most people agreed with David. (O'Reilly, 1996, pp. 71–72).

Not knowing how to go on, O'Reilly ended the lesson and, next day, started in on a new set of activities. Several weeks later, reflecting on that event, she confessed, "It was only after I had acknowledged my difficulties in unravelling the mathematics I was trying to teach . . . that I suddenly began to identify instances where my limitations were getting in the way of my listening and teaching for understanding" (O'Reilly, 1996, p. 73).

In recent years, new standards for the reform of K-12 mathematics education have set an ambitious agenda for mathematics instruction: teaching mathematics for understanding (NCTM, 1989, 1991, 1995). Recognizing that such teaching cannot be a matter merely of presenting math facts and demonstrating algorithms, and that learning for understanding cannot be a matter merely of memorizing those facts and procedures, reform documents envision classrooms in which teachers' and students' roles are drastically revised. Teaching is now seen as a matter of engaging students in significant problems and facilitating discussions about these problems and learning is seen as a process of formulating conjectures, testing out ideas, and exploring alternative approaches. Intellectual authority no longer resides exclusively in teacher and textbook, but is dispersed among members of the classroom community, who offer defensible arguments (Ball, 1993; Cohen, McLaughlin & Talbert, 1993; Confrey, 1990; Lampert, 1988; Mokros, Russell & Economopoulos, 1995).

However, in a practice in which student thinking takes center stage, classroom process becomes much harder to manage and much less predictable (Ball, 1993; Hammer, in press; Lampert, 1988). Indeed, with more and more teachers constructing such a practice for themselves, quandaries such as O'Reilly's are likely to become familiar experiences. As David's response to O'Reilly's question shows, "students' ideas can be as puzzling and oblique as they are inventive and insightful" (Cohen & Barnes, 1993, p. 244). What is a teacher, who is working to establish a classroom culture in which intellectual authority is shared, to do when students propose ideas that are surprising, incorrect, or even, it may seem, incomprehensible?

Whereas dilemmas such as O'Reilly's can never be anticipated in their specificity, teachers can become better equipped to deal with them. If they are to build their instruction around children's thinking, the skills teachers need to acquire include interpreting students' mathematical ideas, analyzing how those ideas are situated in relation to the mathematics of the curriculum, and challenging students to extend or revise those ideas so as to become more powerful mathematical thinkers.

The development of a successful practice grounded in the principles that guide the current mathematics education reform effort requires a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess (Ball, 1989, 1996; Cohen & Ball, 1990; Even & Lappan, 1994; Schifter, 1993; Schifter & Fosnot, 1993; Thompson, A., Phillipp, Thompson, P. & Boyd, 1994). But although there is a growing body of literature documenting the changes teachers go through as they begin to realize the vision of mathematics instruction proposed in the *Standards* (c.f. Fennema & Nelson, 1997; Friel & Bright, 1997; Knapp & Peterson, 1995; Nelson, 1995; Russell & Corwin, 1993; Simon & Schifter, 1991; Wood, Cobb & Yackel, 1991), teachers' developing *mathematical* understandings and how those understandings affect instruction have not received sustained attention from researchers and teacher educators: What kinds of understandings are required of teachers working to enact the new pedagogy? How are these understandings reflected in practice? How can one help teachers learn to think about mathematics in terms of underlying conceptual issues and not merely as an ad hoc sequence of topics?

These are the questions addressed in this paper. Specifically, the paper examines two avenues for promoting teachers' mathematical investigations. The first avenue is *exploration of disciplinary content*, which for K-6 teachers usually involves topics from the elementary curriculum. Although it has often been assumed that elementary-school disciplinary content is simple and that any educated adult knows enough to teach it, this is simply not the case. The mathematics is, in fact, conceptually complex, but those who have been educated in the system needing reform have not dealt with that implicit complexity.

The second avenue is the *examination of student thinking*. A number of projects aligned with current reform efforts have recognized the importance of "student-centered" instruction. Some emphasize teachers coming to see learning as a process of construction (Fosnot, 1989); others teach teachers to conduct interviews of children (Ginsburg & Kaplan, 1988); some present teachers with the findings of researchers who study children's mathematics (Fennema, Franke, Carpenter & Carey, 1993); and still others call attention to students' mathematical talk (Russell & Corwin, 1993; Schifter & Simon, 1992). However, there has been insufficient appreciation of how the analysis of student thinking can, itself, become a powerful site for teachers' further *mathematica*l development.

This paper illustrates how the need arises in K-12 teaching situations for these two kinds of investigations, one related to exploration of disciplinary content and the other to examination of student thinking, and how they can be pursued in a professional development setting. The work described here was conducted in the context of Teaching to the Big Ideas (TBI), a four-year teacher-enhancement project sponsored by the National Science Foundation, and jointly conducted by EDC, TERC, and SummerMath for Teachers (Schifter, Russell, & Bastable, in press). The project involved 6 staff members and 36 elementary teachers. The focus of its work in two-week summer institutes, biweekly after-school seminars, and one-on-one biweekly classroom visits, was the examination of the mathematics of the elementary classroom when teaching is built upon children's thinking.

## INQUIRY INTO MATHEMATICS

"Teachers teach as they have been taught" is a maxim frequently heard. So, too, is its familiar corollary, "In order for teachers to learn to teach differently, they must be taught differently." Widely accepted though these maxims may be, it is not so clear just what their implications for teacher development are. By becoming mathematics students themselves, what can teachers learn about teaching mathematics in new ways? How might professional development opportunities be structured to foster such learning?

In order to address these questions, this paper examines the case of Theresa Bujak,[1] a middle-school teacher. It starts with a look at one lesson in Bujak's sixth-grade classroom, and then steps back in time to consider some TBI mathematics exercises in which she participated. What is the mathematics at issue in Bujak's class and how do her understandings come into play? What aspects of her learnings as a student does Bujak bring back to her teaching?

### *Sixth-Graders Confront a Question About Fractions*

Theresa Bujak teaches in a small school district in a rural community. These days, she organizes her teaching so that most class sessions involve students working in pairs on word problems and then presenting their solutions to the whole group. A question period follows each presentation as Bujak and her class work together to understand presenters' thinking. The atmosphere is friendly and supportive, with students intent upon understanding one another. Students' questions about strategies and representations are seen as efforts to make sense of the ideas offered, not as indicators of poor work or mistaken results. Whereas right answers are valued, much class time is spent sharing methods and diagrams. Conversations about problems continue long after solutions are known.

Bujak's unit on fractions consisted of a series of problems designed to provide her students opportunities to make sense of this new kind of number. According to Bujak,

> Fractions and decimals for sixth graders are often counter-intuitive. The whole numbers that the students have recently become comfortable with (or are still not entirely comfortable with) are now used in different ways and mean different things. Getting comfortable with fraction and decimal numbers and developing a "fraction/decimal" number sense is an important aspect of our mathematics classes this year. (Reflection paper, 2/16/95)

For example, in a lesson (observed 12/12/94 by a TBI staff member, not the author) several weeks into explorations of fractions, Bujak posed this task:

> Solve the following problem by drawing a large, detailed, labeled picture. Write a number sentence (equation). Write the answer in a complete sentence.
>
> > Nancy has 6 2/3 meters of material. It takes 5/6 of a meter to make her fabulous fancy hair ribbons. How many fabulous fancy hair ribbons can she make?

After distributing the worksheet, Bujak reminded her students to work with their partners. In a few minutes the class was fully engaged. Some pairs were trying to draw accurate diagrams using graph paper and rulers; others drew free-hand. The students decided fairly quickly the answer was 8 and spent most of their work time refining their pictures and explanations.

The work of Kathleen and Elizabeth is typical. Kathleen drew seven rectangles and shaded in two-thirds of one of them. See Figure 1(a).

> Kathleen: These are the six things and the two-thirds. Okay?
>
> Elizabeth: Yes. And now we make six in each.

They drew lines dividing each of six rectangles into six parts. They did not draw any lines in either segment of the seventh rectangle. See Figure 1(b).

> Elizabeth: Five of each makes one.

Elizabeth pointed with her fingers to show that each meter would make one ribbon, with one piece left over.

> Kathleen: Then there are six left and four makes another two. So the answer is eight.

As she talked, Kathleen pointed to the diagram to show she was collecting the leftover piece from each meter and combining those six pieces with the four pieces from the two-thirds meter to make two additional ribbons. Kathleen wrote: "You can make 8 fabulous fancy hair ribbons." They then set to work to draw a neater diagram.

Bujak soon called the class to attention and asked for volunteers to go to the board to show their work. Most children raised their hands. Bujak


*Figure 1.* Kathleen and Elizabeth's representations.


*Figure 2.* Nicholas and Irene's representations.

invited Nicholas and Irene to come forward, and they drew the diagram shown in Figure 2(a).

While the whole group studied it, Nancy asked about the two-thirds piece.

> Nancy: I thought the end was 2/3. If she cut it into three, how can you have four?
>
> Nathan: There are six in the whole thing; 4/6 is the same as 2/3.
>
> Annie: Make the dotted lines so they can see the rest of it.
>
> Bujak (to Irene, who is drawing): Can you make it so we can see how it is 2/3?

Irene added a dotted section to her diagram, as shown in Figure 2(b).

> Nancy: So it's not fourths. It's sixths, because of the missing part.

Nancy's question and the responses it provoked among her classmates illustrate this class' concern with unit – the whole to which the fraction refers. In the rectangle at the far right, one section is 1/6 of the whole meter, but is 1/4 of the segment drawn in unbroken lines. Satisfied with this conversation for now, Bujak and her students moved on to consider the arithmetic statement that matched their work.

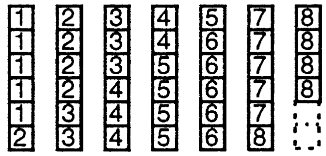


*Figure 3*. Kevin's representation.

Bujak: So what is the sentence?

Annie: 6 2/3 ÷ 5/6 = 8.

Irene: 6 2/3 meters. You need 5/6 meters to make one ribbon. There is 8 of them, so the answer would be 8.

Kevin then came to the board and offered a variation of the diagram already presented, as shown in Figure 3.

Kevin: You have the 6 and you have to divide that up and it makes 7 and you have one left and it – put with the piece of the meter – becomes 8. 6 2/3 is how much material you have. It takes 5/6 to make a ribbon, so 6 2/3 divided by 5/6 tells you how many ribbons.[2]

The class was quiet for a while as they thought through Kevin's comments. Kevin continued by writing his statements on the board: "6 ÷ 5/6 = 7. 1/6 + 4/6 = 5/6 = 1. So it's 8."

Kevin's oral and written statements do describe the process he went through to solve the problem but obscure the issue of the changing unit. Some of his equations are conventionally correct, for example, 1/6 + 4/6 = 5/6. But, for example, his statement, 5/6 = 1, violates rules of conventional notation.

TBI Visitor: I thought 6/6 was 1. How can 5/6 be 1?

Kevin: They are two different things.

Elizabeth: 5/6 is 1 ribbon. 6/6 is just 1.

Kathleen: 5/6 of the material is 1 ribbon.

Elizabeth and Kathleen apparently were able to make sense of Kevin's claims by referring back to the problem context and, so, didn't recognize inconsistencies in his statements. Irene went to the board to explain the situation.

Irene: 5/6 is how much you need to make a ribbon. It's not a whole meter.

She drew the diagram shown in Figure 4 and read her labels aloud.

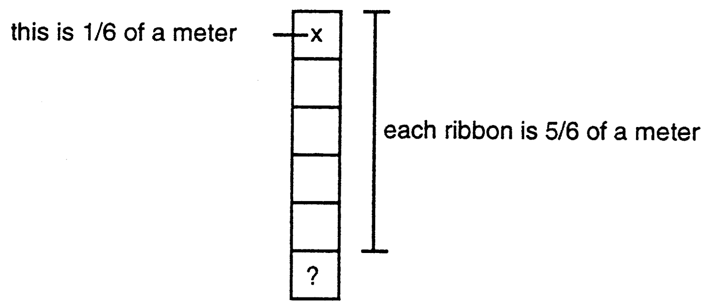


*Figure 4.* Irene's representation of a meter and a ribbon.

The class seemed to accept Irene's labels. However, a disagreement broke out concerning what to call the "extra" piece labeled with a question mark.

Students: It's 1/5 of a ribbon.
No. 1/6 of a meter.
1/6. It's just 1/6, period.

Bujak recognized the significance of this disagreement and invited a few minutes of small-group discussion about what to call that extra piece. While her students debated, Bujak prepared an impromptu demonstration. She pulled out a long strip of paper which she now held in front of her and called the class to attention.

Bujak: This is a meter. Okay?

The students realized what she was doing and joined in by calling out directions to her.

Students: That is one meter.
Now make it in half.
Fold up each piece.

Bujak folded the paper in half and then folded each half into thirds to make sixths, as shown in Figure 5(a).

Students: Now it's in sixths.
Fold one piece back.

Bujak folded back one section of the paper so that the class could now see only the five pieces making one ribbon, as shown in Figure 5(b).

*Figure 5*. Bujak's demonstration with folded paper.

Bujak: So how long is this?

Students: 5/6 of a meter.

Bujak: And this is one ribbon?

Students: Yes.

After establishing that they understood the model, Bujak returned to the original question.

Bujak: What name can I give this section, then?

Nancy: The pieces are still 1/6. There are five of them, but they are sixths.

Bujak: What part of the hair ribbon am I holding onto?

Brenda: One-sixth.

Both Nancy and Brenda were still considering the whole meter as the unit, but Bujak wanted to shake up that conviction.

Bujak: What if I brought this ribbon to the teacher next door?

Irene: Well it is a fifth, if you didn't tell her. I mean, she'd think it was a fifth, if you didn't tell her how we made it.

Bujak: So she would think this section is a fifth?

Frances: You are holding a fifth. There are five equal pieces. That is what it would look like to her.

Brenda: Yeah, but it really is a sixth.

Irene returned to the board and wrote, "One ribbon 5/6 meter. One meter 6/6."

Nathan: One whole hair ribbon is not one whole meter.

Charles: A meter is a hair ribbon and a sixth – no – a fifth of a hair ribbon.

Bujak: For homework tonight I want you to write about this. What names can you give to this section?

By the end of the lesson, Nancy and Brenda were adamant that you must call the section 1/6, while Nathan, Charles, Frances, and Irene were willing to consider that it might depend on which unit you are thinking about, the ribbon or the meter. Bujak would learn from her students' writing where others stood on the debate.

In an after-class conversation, Bujak said the lesson had taken an unexpected turn. She had been hoping the ribbon problem would lead to an exploration of division algorithms and was encouraged by Charles' remarks near the end of class.

Charles was really close to saying that a meter was one-and-one-fifth of a ribbon. This is what happens in division, first one thing is one and then another thing is one. (Consultation, 12/12/94)

She thought she could use Charles's idea to help the class see that both $6\ 2/3 \times 6/5$, as well as $6\ 2/3 \div 5/6$, described the solution.

However, Bujak had recognized that she needed to set her agenda aside, because the students were working on a different, though closely related, issue: how to label fractional pieces. Comments from Nancy, Kevin and Brenda made it clear that the class was working hard to make sense of fraction names. Algorithms for dividing fractions would have to wait.

*From Mathematics Student to Mathematics Teacher*

Initially, readers of this description of Bujak's lesson might be struck by how little was heard from the teacher! Can one even assess a teacher's mathematical understandings, when so much attention is focused on her students? In fact, this episode reveals a great deal about Bujak's mathematics understandings – understandings, furthermore, that were developed in the context of her professional development work.

By Bujak's own account, her mathematics teaching has been changing dramatically. She used to teach the way she was taught in sixth grade, moving page by page through the textbook, explaining the procedures the children were to learn. It was through being a mathematics learner in a new kind of classroom that Bujak began to develop a sense of enlarged possibilities – for mathematics, for herself, and for her students.

A major component of the TBI project involves teachers as mathematics learners in lessons designed by project staff. From as few as four to as many as ten 1- to 3-hour sessions are committed to activities in each of the following areas: number systems and whole number operations, data analysis, geometry, fractions and decimals, ratios, combinatorics, probability, variables, and functions. Even as these topics are listed, it must be noted that such traditional labels do not convey the interconnections and relationships that are highlighted. Once a mathematical area is introduced, it is never left behind, but is rather invoked again and again in other contexts. For example, a unit was set up to allow teachers to explore ratio in the context of geometric shape. Later, learnings from these lessons were recalled and elaborated in work on probability, data analysis, and functions.

A mathematics lesson in TBI generally begins with the introduction of a task, often (and especially for explorations of arithmetic) couched as a word problem. As teachers set to work in small groups, staff members listen in, sometimes posing additional questions to probe and to stimulate teacher thinking. Staff then brings the group together to discuss the task and to use the various groups' findings to carry the mathematical ideas further.

Though these lessons address the content of elementary-school mathematics, they are directed toward mathematical issues challenging to adult learners. Through their work in groups, teachers can identify their past understandings and confusions, pose further questions for themselves and their colleagues, and reach for new levels of understanding; through their journals, they can continue to explore their ideas individually and communicate them on a one-on-one basis with a member of project staff.

For many teachers, participating in these mathematics lessons calls into question their past conceptions of the very nature of mathematics. When challenged at their own levels of competence – confronted with new mathematical ideas – they experience mathematics, often for the first time, as an activity of construction, rather than as a finished body of results to be accepted, accumulated, and reproduced. Thus, as participants explore mathematics content, the mathematics lessons themselves provide grist for reflection: What is the nature of the mathematics you are learning? How is mathematical validity assessed? What does it mean to "think mathematically"?

Furthermore, within the context of such mathematics lessons, many teachers become aware, for the first time, that they, themselves, can be initiators of mathematical thought – that they can offer conjectures, test them out, become curious about mathematical questions, and make their own way through problems they did not expect to be able to solve. Again, their own experience becomes grist for reflection, now concerning the process of learning: What was your experience of putting these ideas together? Can you trace your line of thought? What were the emotions involved? What aspects of the classroom environment promoted or inhibited your learning?

It is in such a setting that Bujak began to develop a sense of the social character of the doing of mathematics. Recognizing herself and her students as initiators of mathematical thought, she realized that she, too, could organize her own classroom around communal inquiry into mathematical ideas. She also knew that, in order to do so, her students would need her help to see mathematics as a social enterprise. Halfway through the year, she observed, “Creating a cohesive group of students who listen to each other and work well with each other has taken time” (Journal entry, 2/10/94).

As Bujak became aware of the power of dialogue to help students make sense of mathematics and of the possibility that limited understanding might lurk behind correct solutions, she developed classroom procedures that would make discussion of her students’ methods and of the basic conceptual issues those methods raised the central experience of engaging the mathematics.

In putting into play such classroom processes, Bujak was making herself vulnerable in the way Anne Marie O’Reilly’s vignette illustrates. By organizing her practice around her students’ mathematical dialogue, she could no longer anticipate when a student might express an idea that would derail her lesson.

Indeed, it is not at all difficult to imagine how a teacher who, having given her class the fabulous-fancy-hair-ribbon problem to work on, might be stymied by debate over what to call Irene’s extra piece – “It’s 1/5 of a ribbon. No, 1/6 of a meter. 1/6, it’s just 1/6, period.” Nor is it difficult to imagine that teacher, flustered by her own confusion, quickly moving to end the discussion. In Bujak’s case, however, her own explorations of fractions in TBI anticipated just this debate among her students, and so allowed her to help them work through their conceptual confusions.

When TBI began its unit on fractions, the group of teachers of which Bujak was a part had already been through an intensive two-week institute and one semester of bi-weekly seminars. During that time, a culture was developed in which teachers understood that in their mathematical explorations, they would be given problems without being shown beforehand how to solve prototypes; it would be their task to think the problems through together, in small groups. They had also come to see that inquiry did not end with a correct solution – one could always dig deeper, make further connections. And they discovered that learning mathematics – doing mathematics – involved posing their own questions to get at the meat of the conceptual issues. Although individual teachers differed in the degree to which they took on these challenges, such were the mores that were coming to define the group’s work together.

When the TBI staff designed the fractions lessons, it was assumed that most teachers’ experiences with fractions had been limited to memorizing the algorithms for computation. (Although some teachers already understood fractions more deeply, the assumption that most did not was grounded

*Figure 6.* Which is more, 3/5 or 3/8?

in a decade's work with teachers.) Thus, lessons were structured to help participants develop tools that would allow them to draw on their own powers of reasoning as they confronted issues new to them.

Most fractions work involved exploring the arithmetic through models of situations from daily life. By examining the quantitative relationships represented in familiar contexts, learners could pursue deeper mathematical insights. There was also emphasis on embodied fractions and their operations with area or linear diagrams and manipulatives. Staff expectation was that, through these activities, the teachers would begin to pose their own questions, make new connections, and confront and work through surprising results.

An underlying theme in all of the lessons involved keeping track of units. Work with both children and adults has shown that as learners move from the realm of whole numbers to that of fractions, the role of unit takes on greater complexity (cf. Hiebert & Behr, 1988; Lamon, 1993; Mack, 1993). "How can that piece of cake be 1/2 and 1/4 at the same time?" is the kind of question frequently heard. Hence, the importance of teachers repeatedly confronting apparently paradoxical results which could only be resolved by sorting out the whole to which the fractions refer.

For example, at the beginning of the fractions unit, when teachers were asked to draw area models to compare 3/5 and 3/8, many came up with the diagram repoduced as Figure 6, which appears to show that 3/5 = 3/8. They were given the opportunity to explore the implications of such a result in order to grasp why comparison of fractions presupposes that they have the same reference whole.

In the next lesson, teachers considered why the following is not an addition problem: *In Maureen's class, 1/5 of the boys are absent and 2/5 of the girls are absent. What fraction of the class is absent*? After all, the analogous problem, *Find the number of absent children in the class if 1 boy and 2 girls are absent,* is solved by addition. Confronted with these questions, the group needed to think through why, when adding or subtracting fractions, the reference whole must stay constant.

In a later session, they were asked to solve such problems as: *Wanda really likes cake. She has decided that a serving should be 3/5 of a cake. If she orders four cakes, how many servings can she make*? When they first diagrammed a solution, as shown in Figure 7, many teachers believed the answer to be 6 2/5. However, the conventional algorithm for solving 4 ÷

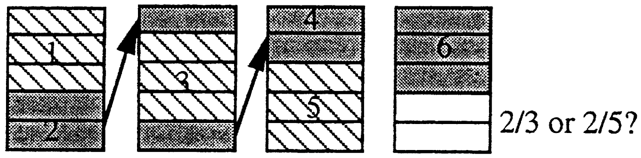


*Figure 7.* A representation of Wanda's cake problem.

3/5 yields 6 2/3. Now teachers needed to consider the different units used for the dividend, the divisor, and the quotient. As teachers worked through these problems, they learned that a great deal of confusion can be resolved by asking: What is the unit to which this fraction refers?

Although in each session questions were raised about units, they were not the exclusive focus. Among the other issues the group worked on were: Why are fractions defined in terms of equal parts? What is really going on in finding equivalent fractions? Why is it said that 1/3 of 15 is a multiplication problem when division is used to solve it? How can it be that a single problem (e.g., *I eat 2/3 of a cup of cottage cheese each day for lunch. I have 2 and 2/3 cups of cottage cheese in my refrigerator. How long will that last*?) can be solved using either addition, subtraction, multiplication, or division?

Like many of her colleagues, Bujak came to the seminar sessions on fractions able to operate with the traditional algorithms. At the same time, she was aware that, although she knew *what* to do to compute with fractions, she did not understand the mathematics behind the sequence of steps. Throughout the seminar she worked to make sense of fractions and operations with fractions in a way that was new to her. In particular, she found that using diagrams to represent problem situations helped her develop a sense of what each operation involved.

I wasn't taught math like this, drawing pictures and visualizing and making models. Now when I think about this work, I see it inside my head. I have a picture. That picture is very powerful for me. [I'm] developing an internal sense of what is going on. (Consultation, 3/12/95)

For example, the seminar problem about Wanda's cakes (see above) provided Bujak the opportunity to examine the mathematics embedded in a procedure familiar to her:

For the first time ever I was able to know where that inverted number in "invert and multiply" came from. . . . When I did repeated subtraction I got an answer of 6 and 2/5 left over. When I did division [applied the invert and multiply algorithm] I got an answer of 6 and 2/3 left over. Every time I looked at the picture to try to explain the different answers I could justify them both. That didn't seem to make sense. And then I saw it!

When I do the repeated subtraction (reality might be cutting the cake for each person that came up and asked for cake) the 2/5 left over is 2/5 of the whole cake left over and that is right. When I do division (real life – cut all the cake and make all the portions at one time) the 2/3 left over is 2/3 of a serving of cake. When I started looking at servings vs. left over cake I then could see the 5/3 (4 divided by 3/5 becomes 4 multiplied by 5/3). The 5/3 is the number of servings in a cake! [Bujak's emphasis.] (Journal entry, 3/15/94)

This journal entry shows Bujak able not only to justify why $4 \times 5/3$ produces the same answer as $4 \div 3/5$, but also to identify actions that would support each interpretation – what she calls "real life" scenarios. Readers can see both the mathematical ideas that Bujak was working on and also the depth of understanding that she was seeking. Until she could articulate how both expressions ($4 \times 5/3$ and $4 \div 3/5$) were represented in a single diagram and connect each arithmetic statement with a set of physical actions, she was dissatisfied with her level of understanding.

It has only taken me how many years to get that invert and multiply thing???? And that is a perfect example of fleeting knowledge and also the fact that it did not impair my ability to DO invert and multiply. I could DO the procedure. This is where the monumental philosophical differences about what it is to "DO mathematics" comes in. I know that there is a difference between doing the invert and multiply procedure and knowing what all the fractions mean. (Journal entry, 3/4/94)

To develop this kind of understanding, to develop an "internal sense of what is going on," now became Bujak's goal for her students – and she employed strategies learned in the seminar to that end:

Insisting that problems be solved using manipulatives, pictures and verbal explanations has been painstaking work. With these things in place, as it seems to be now, everything flows so smoothly. (Journal entry, 2/10/94)

Bujak's objective for the lesson presented in this paper was to begin work on division of fractions with the goal of thinking through the logic of the division algorithm. Thus, she gave her students a word problem with the assignment to come up with a pictorial representation, as well as a number sentence.

At the same time, she realized that a major component of understanding fractions is noticing the unit to which the fraction refers and she worked to be sure her problems would help her students engage with this idea. When asked how she chose the problems she assigned, she reflected, "I have a sense that they [the problems] will promote good discussion. That they would bring up the changing whole" (Consultation, 3/12/95).

Indeed, we can see from this lesson in December that her students had learned to work with word problems and to create diagrams, as well as conventional arithmetic expressions, for use in exploring the mathematics. Following student thinking in this lesson reveals a struggle with the conceptual issue Bujak identified: Given a single meter, 5/6 of which is needed to make a ribbon, what is the name of the leftover portion – 1/5 or 1/6? While some students could see that it is both – 1/5 of a ribbon and 1/6 of a meter – many students could not, feeling sure that it can be only one or the other.

Having wrestled with this issue herself, Bujak recognized the significance of this disagreement and invited the children to talk at their tables for a few minutes about what to call the extra piece. As her students debated, Bujak prepared an impromptu demonstration, pulling out a long strip of paper which she held in front of her and called to their attention. Her demonstration did not resolve the question for her students, but it did clarify what the question was. Nor did she expect that her class would find resolution quickly:

> [The seminar] allowed me to understand that it [takes time] . . . to put this idea together – the changing whole and the division of fractions. It comes up again and again and again. (Consultation, 3/12/95)

Bujak began her fractions unit aware that her students would have to revisit the idea of the changing whole several times before they would feel comfortable with this notion. Even as she sought to develop the connection between problem situation and division-of-fractions algorithms, she was able to register that her students needed work on a prior idea – for, each time she posed a problem which ought to have led to a discussion of computation, they engaged, instead, in debate about labeling fractional pieces. Bujak's acknowledgment of the time it took *he*r to put these ideas together, and her knowledge of the relatedness of these ideas, made such class discussions possible.

Thus, Bujak's actions in this lesson show how she called upon new beliefs about the nature of mathematics and mathematical engagement, and new understandings of specific content and how it is learned. In her meeting with the TBI staff member after this lesson, Bujak reflected on her students' experience:

> It reminded me of what I went through with the cake problem. Because of the changing whole. How can one object have two names? How can one piece be one-fifth and one-sixth at the same time? (Consultation, 12/12/94)

Then she added that she planned to continue the discussion the next day, this time in the context of Wanda's cake problem.

## INVESTIGATING STUDENT THINKING

Bujak's case demonstrates how a teacher's own work as a mathematics learner opens up new possibilities for her. Specifically, her mathematical studies help her see how to organize her teaching around communal inquiry into mathematical ideas. Her studies influence her goals for student learning and the methods she employs to reach those goals, and allow her to recognize the complex mathematical issues her students need to confront. These turned out to be the same issues she needed to confront and work through.

However, it is also possible to identify aspects of Bujak's practice that are less likely to be fostered by her own study of mathematics: the skills of listening to students' words, interpreting the mathematical ideas they express, and identifying the relationship between student thinking and the mathematics on her agenda. Yet, precisely because her students' confusions so closely matched those that Bujak, herself, needed to wrestle with, it is difficult to see these other skills being exercised. For this purpose, a visit to Beth Keeney's classroom is more revealing.

### *Third- and Fourth-Graders Confront a Question About Fractions*

Beth Keeney teaches a combined third- and fourth-grade class in an independent school in the suburbs of Boston. For the last two years, she has organized her mathematics teaching around what she calls the "question of the day" (or "qod"). As the children enter the classroom first thing in the morning, they read the qod written on the white board and, settling in, get out their notebooks. One finds the children scattered throughout the room – lying on the floor, sitting at one of several tables, or slouched on the sofa – working on the problem. Some work together, some work alone – as they choose. The children feel free to ask one another, or any adults in the room, to help them think the problem through.

When Keeney judges that the children have had enough time to get into the problem, she calls them together for a group discussion. Children and teacher sit in a circle on the floor to share their thoughts.

In addition to the qod, Keeney employs other devices to support the development of her students' mathematics. The children are given individual weekly assignment packages (for mathematics as well as other subjects) which they work through during daily "independent" periods. At times she organizes small-group problem solving, each group working on a set of problems specifically selected by Keeney to challenge the group at its level.

To get a sense of what Keeney values in her students' mathematical engagement, consider the following excerpt from her journal:

I . . . watch Betsy and Evan while they work and note the tremendous amount of attention they give their work, the extraordinary ability they have to lean into the discomfort of thinking. There are a number of children in the class who exhibit this same power. . . . These children are successful thinkers. . . . It is significant to me that persistence is so crucial to creative and deep thinking in math. Look at Betsy's "question" for the second part of her qod. She spent nearly 40 minutes writing this question. There are a lot of ingredients to her final product: persistence, facility with language, freedom of time, a way of knowing (and I don't really know what I mean by that except to say that there is an inner dialogue that goes on for these children; there is someone inside them who communicates with their questioning self) and of course a fundamental notion that number is intrinsically connected with language and idea. (Journal entry, 4/5/94)

Keeney's own recent work as a mathematics student has taught her the value of "leaning into the discomfort of thinking." She, too, has persisted – at times through tears – learning to tolerate the frustration of confusion when seeking clarity. She now thinks of mathematics as ideas that can be deepened and extended as she engages with her colleagues, or even her own questioning self. Having brought this knowledge of what mathematics can be into her own teaching, she now recognizes the power of her students' mathematical process.

Keeney began working on fractions with her students in March, while she was studying fractions in the TBI seminar. Although she had taught this age group the previous year, her new perspectives on mathematics made her feel as if she were venturing into uncharted territory. What did her students know about fractions and what did they need to learn? These were the questions she posed to herself as she opened up the topic to her class. They were not questions she expected a textbook or resource guide to raise. Rather, she was looking for the major conceptual issues that these children needed to work through in order to develop as mathematical thinkers, and she expected to find them by listening to her students. A few days into the unit, she reflected in her journal:

They have been "doing fractions" since they started sharing things and it has been a minimal but integral part of conversations for a long time – since two years old maybe. But I am finding that they know next to nothing about how to express their ideas about smaller than one. They haven't any information about fractions smaller than 1/2 or 1/4 for the most part and so all this stuff is brand new and intriguing to me. (Journal entry, 3/21/94)

The following dialogue occurred shortly after Keeney's class initiated their fractions unit, in the middle of a discussion, on a day (3/23/94) on which a TBI staff member (the author) was visiting. Although the lesson began, as usual, with the qod, the whole-group discussion quickly ventured away from its particulars:

Jonathan: I sometimes don't understand the word "fraction."

> Keeney: Yeah.
>
> Jonathan: I think it seems to be hard to explain.
>
> Keeney: Because you've asked people and they haven't been able to give you a very clear idea of it?
>
> Jonathan: Right.
>
> Keeney: Let's ask everybody. Betsy, what is a fraction? You were going to give a definition. And Kyle, maybe if you think about this, you might want to contribute also to what is a fraction. Betsy?
>
> Betsy: Well, I think of a fraction as less than a whole. It isn't a whole; it isn't a half, and it isn't a quarter, usually.

The discussion continued, with various thoughts about what fractions are. For example:

> Jonathan: Sometimes the way I understand the word fraction is different parts. Like two fractions would be a half. Four fractions would be a quarter. . . .
>
> Kyle: Well, if you think about a fraction – anyway, it's not a whole but it's something of a whole. Let's say it's one fourth of 20. Well, it's 5, because you have 5, 10, 15, 20. And you're dividing 20 into 4, and [it's] one of those sections, one of those groups. . . .

Harriet got out some cubes and arranged six of them into three groups of two.

> Harriet: A fraction is when things are split up into even groups. So I've now arranged them into three groups. So you know what this fraction would be called?
>
> Jonathan: Three fractions?
>
> Harriet: No. It would be called a third. . . .
>
> Jonathan: Why are they called a third?
>
> Harriet: Because there are three of them.
>
> Jonathan: Why do you call it a third? I mean, like, uh, it could just as well be three quarters.

Still later in the discussion, Jonathan posed his question another way:

> Jonathan: The question I'm asking is if things are a fraction, why don't they say the thing instead of a fraction?

Along with Keeney, the visiting TBI staff member listened to the conversation, both of them trying to sort out different students' understandings. And along with Keeney, she was working hard to decipher just what Jonathan was asking. Some minutes later, after the class had returned to the qod to consider a proposed answer, 14/24, the staff member offered a question:

> TBI visitor: The fraction was 14/24. And so the question is, sort of, what's the difference between the answer, 14, and the answer, 14/24?
>
> Jonathan: Right.

In the consultation between Keeney and the staff member after the lesson, Keeney began to identify the idea her students needed to "wrap around":

> What a new place it is for them to think of a fraction as an idea. Which is why when you were talking about, towards the end of our meeting, about the 2 and the 1/3 [or 14 and 14/24] . . . – for these kids to sort of wrap around that thought, that it's an idea – [that a fraction is] a really different thing than whole number. (Consultation, 3/23/94)

Most of the children in the class still needed to sort out the notion that their classmate, Kyle, began to articulate: "It's not a whole but it's something of a whole." Given Harriet's arrangement of 6 cubes into 3 groups of 2, the children needed to think of the 2 cubes not only as 2, but also as 2 in relation to the 6. And at the same time that each cube is a unit, so do the 6 cubes make up a unit, and the 2 cubes make up 2/6 or 1/3 of that unit, or that one whole.

Recognizing the importance of this issue, Keeney decided to structure her qods specifically to point the class toward it. When the staff member returned a few days later, Keeney had presented her students with the following problem:

> Maria, Angelo, and Christina were sharing a giant round tortilla covered with queso (cheese) and salsa. The tortilla was divided into 12 slices. Write a question for which the answer is 3. Write a question for which the answer is 3/12. *Bonus: Write a question for which the answer is 6/24. (Observation, 4/1/94)

As children worked on their own, the staff member heard Jonathan ask a classmate, "What is 3/12? What does it mean?" And he brought the question up again in the whole-group discussion:

> Jonathan: I still don't get what 3/12 is.

Keeney: Right . . . But you know what, Jonathan? I read a lot of questions that people wrote in response to this exact problem – when do you get to call it "3," and when do you get to call it "3/12"? And there are a large number of people in here who have the exact same question, but haven't figured out how to phrase it just the way you did. Which is, "I don't understand where '3/12' comes from. I don't know why you name it '3/12' as opposed to '3.' "

Jonathan: Yeah.

Keeney: Okay. This will continue to be a question for you. And I want you to know that it is a question for a lot of people in here.

Before continuing the discussion of fractions, Keeney asked the children to share the questions they wrote whose answer is 3. Very quickly, however, their attention turned back to the meaning of 3/12.

Adam: 3/12 is 3 out of 12, not just 3. 3 is not like out of anything. It's just a whole number.

Keeney: So you could . . . open the refrigerator door. And you can find 3 oranges in there. But if you went to the market and you bought a dozen oranges and you ate 3 of them on the way home, that would be 3 out of 12 as opposed to just 3 oranges?

Adam: Yep.

Carla: 3/12.

Keeney: That would be 3/12?

Joan: If you split them up, that's a fraction.

Keeney: If you don't have to split them up, you leave them as whole numbers?

Jonathan: But if you take it from the market and you buy it, and you have 3 of them, that's 3 oranges, and there could be more oranges at the market.

Fred: So that could be 3 out of 12. . . .

Harriet: You took 12 from the market and you eat 3 out of those 12; it's 3/12.

Carla: The 12 that you buy is the whole number of oranges. And then the 3 out of the 12 oranges is the fraction. Say you bought the 12 oranges at the market. So you've got 12 oranges. And on the way home you eat 3 of them. Now erase from your mind all the other oranges in the market.

Jonathan: That would be zero.

Carla: No. . . . You keep the ones in the car that you have, but erase from your mind the others. They don't mean anything. Then you eat the 3 from the 12, and the other oranges at the market don't have anything to do with what you need to find out.

Jonathan: But that still doesn't answer my question.

Keeney: You know what I would like – something that I noticed, Jonathan, about what you're talking about. Is that the 3 oranges that you eat are the same 3 oranges that you eat whether there's 3 of them or 3/12 of them.

Jonathan: Yeah.

Keeney: They're the same oranges. And so we're talking about the next level of how to describe them. But you're absolutely right. There's no difference between 3 oranges and 3 oranges.

In the days between the staff member's visits, the children's ideas about fractions had continued to develop. Different components of the concept were coming together for different children, and the explanations offered about what fractions are were becoming more sophisticated. However, Jonathan, for one, still seemed stuck on the same question: Why call it "3/12" when you can see it's 3?

Keeney appreciated Jonathan's willingness to be public with his question and his attempts to articulate his confusion. For it was with Jonathan's help that she was able to identify a major conceptual hurdle her students needed to work through. In order to track the development of various students' conceptions, she decided to write a set of qods similar to the "giant round tortilla" question (e.g., *Ana and Jose were sharing a pizza. The pizza was divided into 8 slices. #1) Write a question for which the answer is 2. #2) Write a question for which the answer is 2/8. Challenge: #3) Write a question for which the answer is 4/16*). She gave the class different versions of the same problem several days apart and sometimes collected the children's written work, carefully studying their responses in order to understand just what they were thinking; other days, she listened to the whole-group discussion to follow how the children's ideas were developing. After several weeks, she shared her observations:

[There's an idea that's] sort of floating through the classroom. [There are] kids who weren't even really aware that [Jonathan's question] was a question. . . . Last week it was Keith and Harriet and Maura who hadn't formulated the idea that 2 and 2/8 would be different ideas. . . . My understanding is that Maura is probably much clearer on the defining factors of those things. Carla is certainly much clearer. And Adam knows what's going on, and there are a few other children who [do]. And now there's this whole other crop of kids who have decided that 2 and 2/8 are the same. It's sort of like they're a week behind. They didn't really have even enough going to make a decision about it last week, to even really make a decision about the fact that 2 and 2/8 were the same or weren't the same. And Andy is one of them and I think he's now saying, the question is good enough for both of those answers. And so he's sort of a week behind the last crew, but it's really interesting to me to note that he wasn't even aware of this dilemma at all last week, that Jonathan was way ahead of him in even asking the question. Paul hasn't even gotten to this decision that they're the same thing yet. . . . It's like there's this whole group of kids who are sort of having the same community conversation. And yet, they are definitely coming at it from whatever they have in their own little suitcase – and hearing pieces of it, and not hearing other pieces of it, and coming the following week to this same thing. (Consultation, 4/13/94)

Although Keeney had identified a major conceptual issue her students needed to work through, she knew that she could not understand it for them. She *could* work to create situations in which the children would have opportunities to confront it. As she provided such opportunities, she watched closely to see which children seemed to have moved through that issue; which children were turning to face it, but hadn't yet found their way through; and which didn't even realize it was there. Her classroom environment was respectful of the children wherever they happened to be, but she saw it as her challenge to keep them all actively thinking and learning, building stronger mathematical conceptions.

*From the Seminar to the Classroom: Learning to Listen in New Ways*

As with Theresa Bujak, so, too, did Keeney bring to her classroom a new understanding of what it means to learn and to teach mathematics. Although the notion that mathematics is about ideas was not new to Keeney, the possibility of teaching to mathematical ideas was. In fact, one year into TBI, she came to view engagement with ideas as the point of her mathematics teaching:

I'm convinced that what was missing from the qod last year was talking about numbers as a whole way of [thinking]. . . . This year has much more to do with the behavior of numbers and . . . last year it had to do with answers to the questions. . . . To engage in strategy alone – which is, I think, how I thought of math and the qod [last year] . . . that's like a dead end. I mean, it's certainly worth doing and kids may try somebody else's, but at some point there's not much there. . . . And so I think to have something to really mull over – a dilemma – is really much more interesting. (Consultation, 4/1/94)

In her own work as a mathematics student, Keeney had already learned that mathematics is not merely a set of procedures to be memorized and mechanically applied. Rather, she – and her students – could confront problems never seen before and apply their ingenuity to work them through. Furthermore, she was learning that mathematics was more than solving novel problems and sharing solution strategies. Instead, the problems had become a means to exploration of mathematical concepts.

However, the possibility of engaging children in an exploration of ideas presupposes being able to identify the concepts that are at issue for them. But this is no simple task, because their ideas are necessarily expressed in the words of young beginners. Furthermore, in contrast to the issues surfaced in Bujak's lesson, the perplexities that Keeney's students needed to work through had been faced, if at all, by their teacher when she was a child, and had not been raised to consciousness since.

As Jonathan presented his confusions – "Why are they called a third?" "Why don't they say the thing instead of a fraction?" "I still don't get what 3/12 is" – Anne Marie O'Reilly's predicament is again called to mind. For O'Reilly, David's surprising interpretation of the fraction 1/2 was a lesson stopper. Today (four years later), O'Reilly is prepared – as was Keeney with Jonathan's puzzlement over the very need for fractions – to dig under her children's words to find the sense in their perplexities.

Retrieving a child's perspective on a concept, many years after one's own initial acquaintance with that concept, can offer powerful mathematical insight. Aspects of an idea, perhaps long since buried, perhaps never before noted, can come into view. For example, adults are generally unable to recall the time when their concept of number was confined to the experience of counting whole numbers. Yet, listening to children being introduced to the idea of fraction, and realizing how this challenges their very notion of number, offers adults an opportunity to think through how the concept of number expands as one moves from the domain of whole numbers to that of fractions. It is no longer a matter merely of counting units. Instead, one must now count the number of units in one quantity, count the number of units in a second quantity, and derive a third number – a new kind of number – that indicates the first quantity in relation to the second.

Though most adults, including most elementary teachers, learned years before how to operate with fractions, few developed an articulated sense of the conceptual distinctions between whole numbers and fractions – something that Keeney's students were now working towards. Thus, attempts by Keeney and her TBI colleagues to make sense of Jonathan's question – "Why don't they say the thing instead of a fraction?" – and to come up with answers for themselves, deepened their own understandings of what fractions are and how they differ from whole numbers.

An important part of the work of the TBI seminar involved learning how to use children's perplexities to make such deep conceptual issues visible. The TBI teachers began their investigations into children's mathematical thinking by analyzing other teachers' students, studying videotapes of clinical interviews and classroom discourse as well as written materials illustrating student work. These analyses were supplemented with readings of assigned research articles. For example, during the fractions unit, a videotape from Deborah Ball's classroom depicted her third-grade class discussing which is larger, 4/4 or 4/8. Accompanying documentation included excerpts from notebooks in which students reflect on their classmates' conjectures and arguments as well as from their teacher's journal (Ball, 1990). Using this rich set of materials, teachers analyzed how Ball's eight- and nine-year-olds confront the idea of the reference whole of a fraction.

At the same time, assigned readings raised similar issues. "Magical Hopes: Manipulatives and the Reform of Math Education," by Ball (1992) challenges the commonly held belief that manipulatives provide the key to promoting students' mathematical understanding. To make the point, Ball describes how Jerome puzzles over fraction bars to address his teacher's question, "Which is more – three-thirds or five-fifths?" "Jerome is struggling to figure out what he should pay attention to about the fractions models – is it the number of pieces that are shaded? the size of the pieces that are shaded? how much of the bar is shaded? the length of the bar itself?" (p. 17). "Halves, Pieces, and Twoths: Constructing Representational Contexts in Teaching Fractions," also by Ball (1993) offers, first, a short overview of the research literature and how it connects to planning a third-grade unit on fractions, and then takes readers into her third-grade classroom to explore how representational contexts are co-constructed by members of the class and their teacher. In "Of-ing Fractions," Moynahan (1996) describes her sixth-graders' attempts to decide which arithmetic sentences model such problems as, "1/3 of the 15 people at the picnic were Davises. How many Davises were at the picnic?" Moynahan's paper goes on to raise further questions about the relationship between a mathematical representation and the situation it models, as well as about what constitutes a valid argument.

However, the teaching practice TBI was working to help these teachers construct requires a facility that goes beyond an understanding of basic principles of children's mathematical thinking in general. Rather, it involves developing a new ear, one that is attuned to the mathematical ideas of one's own students. This latter point is expressed in the words of teachers as they began to listen to their students in new ways:

> I listened for right answers, confirmation that the students understood what they had been taught. I was accustomed to listening for specific indicators that a student was following my line of thinking.

It is almost as if I heard them previously, but I had my next statements already planned. I attempted to adjust their thinking to what I planned to say next, instead of analyzing what they said to determine what I should ask, say, or do next.

In a practice that puts students' ideas at the center of instruction, teachers now listen to students not only to assess the extent to which those students' ideas match their own, but also to understand these ideas in their own right. What is it that this child is communicating? What is the sense in this child's idea? What does he/she understand? What is the child confused about? What is the issue that the child is working on?

I'm [becoming] able to see how individual kids are thinking and see what concepts are troublesome for kids to make sense of. . . . I feel like I'm getting more skilled at finding out what kids do "get," rather than just thinking "they don't get it."

Within TBI, teachers' own classrooms became a major resource for learning about student thinking. One mechanism developed for such investigations is "episode writing." Twice in the first year, and as a regular monthly assignment in the second and third, teachers wrote scenarios – episodes – from their own teaching. The assignment was to write a 2- to 5-page narrative that captured some aspect of the mathematical thinking of one or more students, using transcriptions of classroom dialogue or samples of students' written work.

Equipped with illustrations from their own classrooms, teachers met in small groups during seminar sessions to share and discuss their work, explaining what they saw in one another's episodes, and airing their own questions. Episodes were then turned in to project staff, who worked to figure out what was revealed about student thinking, and who used their written responses to highlight this for teachers. In general, staff did not comment on the teaching, but posed questions about the ideas the students were working on.

Finally, members of the staff selected sets of episodes to bring back to the whole group in order to look for themes common across classrooms. For example, the seminar might group three episodes, each telling a story about a child, or group of children, working to sort out base-ten relationships while exploring ways to add multi-digit numbers. Or it might consider a set of episodes in which groups of children are solving problems that involve adding fractions. During the third year of the project, teachers were assigned to focus groups that met over the course of the semester to concentrate episode writing and discussion on particular topic areas.

In addition to the episodes, staff visits to participants' classrooms provided another mechanism for investigating children's thinking. These visits took place twice each month during the first two years of the project. The pedagogical (as opposed to research) purpose of these visits was to provide support to teachers as they worked to develop a classroom culture promoting significant mathematical activity and dialogue. Each session consisted of a classroom visit followed by a half-hour one-on-one discussion. At the beginning of the year, teachers chose their own goals as a starting point for transforming their instruction. During the half-hour discussion, visiting staff members did not focus on classroom logistics, but worked to pose questions that would continue the process of reflection: What do you want your students to learn? How will you find out what they already know? What activities will you choose and how will those activities help your students learn? How will you know what they have learned? As teachers became oriented to their new practice, the one-on-one discussions moved toward close study of student thinking: Teacher and staff analyzed dialogue and written work together, looking for the big ideas students were confronting.

Keeney's activity during her fractions unit shows what is involved as teachers learn to listen to their students' mathematical discussions, interpreting their words to identify the issues they are working on, and choosing tasks to support their development. At the start, she was unclear about what her students needed to learn, but she did know it was her task to figure that out as she attended to their discussions. And so, she put out questions for the class to work on and then listened hard.

In the articles assigned TBI teachers, Keeney had read about young children working to make sense of what a fraction is. For example, she read about how Jerome, inspecting that fractions bar, didn't know if he should be paying attention to the number of pieces shaded, the size of the pieces shaded, the amount of the bar that is shaded, or the length of the bar, itself (Ball, 1992, p. 17). However, as she listened to her own students, it was not clear to her how their questions related to what she had read or, indeed, her own understanding of fractions.

When one is confronted by Jonathan's third-grade articulations of his question – "Why do you call it a third? I mean, it could just as well be three quarters." "Why don't they say the thing instead of a fraction?" – it is not immediately obvious just what he is asking. But by considering his remarks in the context of his classmates' demonstrations and arguments, as well as of her readings about children's mathematical thinking, Keeney, with the help of her TBI visitor, identified what seemed to be at issue for him and his classmates. And as the two adults probed further, it appeared that the difficulty lay with the idea that a fraction represents a given quantity *in relation to* another.

Although the children's conceptual issues were not the same as those the teachers in the seminar worked through, Keeney was reminded of a problem posed to her colleagues: *Jorge has 2 pizzas, one pepperoni and one cheese. Each pizza is cut into 8 slices. He eats 1 slice of the pepperoni and 2 slices of the cheese pizza. Write a word problem about Jorge's pizza eating so that the answer is 3. Write a word problem about Jorge's pizza eating so that the answer is 3/8. Write a word problem about Jorge's pizza eating so that the answer is 3/16*. And so, she used variations of this problem, for example, the giant tortilla problem, as her assessment tool. In examining written responses to the giant tortilla problem, Keeney considered the children individually: which were struggling with the difference in meaning between 3 and 3/12? Which had largely worked it through? Which were not yet ready to approach it? By giving them similar problems over the next few weeks, she could track how their conceptions developed.

After the fractions unit ended, Keeney remarked that she expected her fractions lessons next year to be better, that it would be much easier "to go deeper into it earlier on for kids. . . . I do think it will have made a difference next year doing fractions, obviously knowing much more about them myself." And when asked about how her learning took place, she said that it was through both doing mathematics herself and analyzing student thinking.

> The point of view of someone who has never been taught how to do all of that stuff really enriches my own thinking about fractions. And, of course, will show me much more where the stumbling blocks are or where the things to trip over are – more than my own work in math. So I'm thinking for me it's very, very important for both of those things to be happening at the same time. (Consultation, 5/4/94)

Particularly important, she said, was the careful analysis she did of the children's written responses to two of her qods. "I needed . . . not just the listening to what they were doing, but I really needed to look at and think about their answers."

## CONCLUSIONS

This paper opened with a scene from Anne Marie O'Reilly's classroom. Stumped when one of her students offered an unexpected and puzzling idea, she didn't know how to respond. The next day, she reported, "I decided to put the geoboards aside for a while and have the students try some activities with fraction strips" (O'Reilly, 1996, p. 72).

Teachers who choose to develop a practice like the one to which O'Reilly aspires will almost certainly face similar situations. When teaching is designed to elicit student thinking, there are likely to be surprises.

However, as O'Reilly (1996) herself has pointed out, professional development projects can help teachers become better prepared to confront them. With a richer understanding of the domain she and her class were broaching, O'Reilly might have recognized other, more productive options than changing the subject. With more experience listening to students in order to sort out the mathematical issues they are working on, she might have been able to pose additional questions to David and his classmates, probing to understand how their thinking connected with conventional meanings for fractions. Indeed, in the four years since O'Reilly originally reported the vignette told here, she has continued to take advantage of professional development opportunities and reports that she can engage her students in mathematical discourse with much greater confidence and fluency.

In this paper, the cases of Theresa Bujak and Beth Keeney have been used to illustrate how teachers call upon learnings from a professional development program – learnings about mathematics and about children's mathematical thinking – as they engage their students in a study of fractions. Indeed, the two cases indicate quite specifically which aspects of their professional development experiences proved most transformative for their practice.

In explorations of disciplinary content, Bujak, Keeney, and their colleagues developed new conceptions of the nature of mathematics and a heightened sense of their own mathematical powers. As learners of mathematics, they experienced a new kind of classroom, one in which the social character of doing mathematics was realized, and in which student thinking took center stage. In addition, these mathematics lessons became occasions for teachers to reflect on their own process of learning and to consider those features of the classroom that supported or hindered that process. All of these inspired teachers to envision a new kind of practice for themselves and helped to prepare them to put it into play. Finally, the fact that the mathematics content under study in this professional development setting is the very content they teach prepared them to engage their students more responsively and with greater mathematical fluency.

Parallel investigations into student thinking allowed TBI participants to analyze the resources children bring to the content of their learning, assess the conceptual challenges they face, and identify the sorts of confusions that typically arise. Published cases and articles about, and videotapes of, other educators working with children provided important opportunities for discussion and exploration. However, teachers also needed to learn to listen and to hear their own students making sense of mathematics. To this end, writing episodes – two- to five-page vignettes, capturing the thinking of one or more students – proved an effective medium. The exercise permitted the kind of close examination of their students' words that helped teacher-authors work out which understandings, questions, and confusions lay behind those words.

Although the activities of the TBI seminar were intended to help participants work through particular content areas – with the dual focus of engaging in the mathematics for themselves and examining children's mathematical thinking – they were also meant to help the teachers develop a disposition to inquiry. For the new understandings they must develop and the teaching situations they must negotiate are too varied, complex, and context-dependent to be anticipated in one or even several courses. Teachers must become investigators – of mathematics and of student thinking – in their own classrooms (Ball, 1996; Heaton, 1996; Featherstone et al., 1993; Russell et al., 1995).

As Keeney's fractions work with her students exemplifies, this imperative is not simply a matter of remediation. Rather, it is an aspect of the very nature of the mathematics of the "reformed" classroom. For when mathematics is understood as a network of connections among various concepts, their representations, and the contexts in which they are embedded, no piece of mathematics is ever fully understood; with any mathematical concept, there are always new ways to look at it and more connections to be made. Similarly, although there are patterns in children's engagement with particular mathematical concepts, the individual experiences that a single child or group of children may bring to a particular question are unlimited. Thus, teachers must come not only to expect, but to seek, situations in their own teaching in which they can view the mathematics in new ways, especially through the perspectives that their students bring to the work.

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## NOTES

[1] The names Theresa Bujak and Beth Keeney are pseudonyms, as are the names of the children in their classes. Anne Marie O'Reilly's vignette has been published under her own name. Her students are identified by pseudonyms.
[2] The staff member present, who saw Kevin's gestures and noted his pauses and intonation, interpreted his words as follows: "You have the 6 [meters] and you divide that up and it makes 7 [ribbons]. And you have one [small section] left [from the 6 meters] and [you] put [it] with the [2/3] piece of the meter. [Now it] becomes 8 [ribbons]. 6 2/3 [meters] is how much material you have. It takes 5/6 [meter] to make a ribbon, so 6 2/3 divided by 5/6 tells you how many ribbons."

## REFERENCES


Ball, D.L. (1989). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophey (Ed.), *Advances in research on teaching: Vol. 2. Teacher's subject matter knowledge and classroom instruction* (1–48). Greenwich, CT: JAI Press.

Ball, D.L. (1990). *Fractions: Excerpts from Deborah Ball's third-grade class, May 1990.* East Lansing, MI: Mathematics and Teaching through Hypermedia.

Ball, D.L. (1992, summer). Magical hopes: Manipulatives and the reform of math education. *American Educator*, 16(15–18), 46–47.

Ball, D.L. (1993). Halves, pieces, and twoths: Constructing representational contexts in teaching fractions. In T.P. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (157–196). Hillsdale, NJ: Lawrence Erlbaum Associates.

Ball, D.L. (1996). Connecting to mathematics as part of learning to teach. In D. Schifter (Ed.), *What's happening in math class, Volume 2: Reconstructing professional identities* (36–45). New York: Teachers College Press.

Cohen, D.K. & Ball, D.L. (1990). Policy and practice: An overview. In D.K. Cohen, P.L. Peterson, S. Wilson, D. Ball, R. Putnam, R. Prawat, R. Heaton, J. Remillard & N. Wiemers (Eds.), *Effects of state-level reform of elementary school mathematics curriculum on classroom practice* (Research Report 90-14), 1–8. East Lansing, MI: The National Center for Research on Teacher Education and The Center for the Learning and Teaching of Elementary Subjects, College of Education, Michigan State University.

Cohen, D.K. & Barnes, C.A. (1993). Conclusion: A new pedagogy for policy? In D.K. Cohen, M.W. McLaughlin & J.E. Talbert (Eds.), *Teaching for understanding: Challenges for policy and practice* (240–276). San Francisco: Jossey-Bass Publishers.

Cohen, D.K., McLaughlin, M.W. & Talbert, J.E. (Eds.) (1993). *Teaching for understanding: Challenges for policy and practice*. San Francisco: Jossey-Bass Publishers.

Confrey, J. (1990). What constructivism implies for teaching. In R.B. Davis, C.A. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics, Journal for Research in Mathematics Education Monograph No. 4* (107–124). Reston, VA: National Council of Teachers of Mathematics.

Even, R. & Lappan, G. (1994). Constructing meaningful understanding of mathematics content. In D. Aichele & A. Coxford (Eds.), *Professional development for teachers of mathematics: 1994 Yearbook NCTM* (128–143). Reston, VA: National Council of Teachers of Mathematics.

Featherstone, H., Smith, S.P., Beasley, K., Corbin, D. & Shank, C. (1993). *Expanding the equation: Learning mathematics through teaching in new ways*. Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.

Fennema, E., Franke, M.L., Carpenter, T.P. & Carey, D.A. (1993). Using children's knowledge in instruction. *American Educational Research Journal*, 30, 555–583.

Fennema, E. & Nelson, B.S. (Eds.) (1997). *Mathematics teachers in transition*. Hillsdale, N.J.: Lawrence Erlbaum Associates.

Fosnot, C.T. (1989). *Enquiring teachers enquiring learners: A constructivist approach for teaching*. New York: Teachers College Press.

Friel, S. & Bright, G. (Eds.) (1997). *Reflecting on our work: NSF teacher enhancement in K-6 mathematics*. Lanham, MD: University Press of America, Inc.

Ginsburg, H. & Kaplan, R. (1988). *Overview: Math video workshop*. New York: Teachers College, Columbia University.

Heaton, R. (1996). Learning while doing: Understanding early efforts to create new practices of teaching and learning mathematics. In D. Schifter (Ed.), *What's happening in math class, Volume 2: Reconstructing professional identities* (74–80). New York: Teachers College Press.

Hammer, D. (in press). *Discovery learning and discovery teaching*. Newton, MA: Center for the Development of Teaching, Education Development Center, Inc.

Hiebert, J. & Behr, M. (1988). Introduction: Capturing the major themes. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (1–18). Reston, VA: National Council of Mathematics.

Knapp, N.F. & Peterson, P.L. (1995). Teachers' interpretations of "CGI" after four years: Meanings and practices. *Journal for Research in Mathematics Education*, 26, 40-65.

Lamon, S. (1993). Ratio and proportion: Children's cognition and metacognition. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (131–156). Hillsdale, NJ: Erlbaum.

Lampert, M. (1988). The teacher's role in reinventing the meaning of mathematics knowing in the classroom. In M.J. Behr, C.B. Lacampagne & M.M. Wheeler (Eds.), *Proceedings of the tenth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (433–480). DeKalb, IL: Northern Illinois University.

Mack, N. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (85–105). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mokros, J., Russell, S.J. & Economopoulos, K. (1995). *Beyond arithmetic: Changing mathematics in the elementary classroom*. Palo Alto, CA: Dale Seymour Publications.

Moynahan, J. (1996). Of-ing fractions. In D. Schifter (Ed.), *What's happening in math class, Volume 1: Reshaping practice through teacher narratives* (24–36). New York: Teachers College Press.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

Nelson, B.S. (Ed.) (1995). *Inquiry and the development of teaching: Issues in the transformation of mathematics teaching*. Newton, MA: Center for the Development of Teaching, Education Development Center, Inc.

O'Reilly, A.M. (1996). Understanding teaching/teaching for understanding. In D. Schifter (Ed.), *What's happening in math class?, Volume 2: Reconstructing professional identities* (65–73). New York: Teachers College Press.
Russell, S.J. & Corwin, R.B. (1993). Talking mathematics: "Going slow" and "letting go." *Phi Delta Kappan*, 74, 12.
Russell, S.J., Schifter, D., Bastable, V., Yaffee, L, Lester, J.B. & Cohen, S. (1995). Learning mathematics while teaching. In B.S. Nelson (Ed.), *Inquiry and the development of teaching: Issues in the transformation of mathematics teaching* (9–16). Newton, MA: Center for the Development of Teaching, Education Development Center, Inc.
Schifter, D. (1993). Mathematics process as mathematics content: A course for teachers. *Journal of Mathematical Behavior*, 12, 271–283.
Schifter, D. & Fosnot, C.T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform*. New York: Teachers College Press.
Schifter, D., Russell, S.J. & Bastable, V. (in press). Teaching to the big ideas. In M. Solomon (Ed.), *The diagnostic teacher: Revitalizing professional development*. New York: Teachers College Press.
Schifter, D. & Simon, M.A. (1992). Assessing teachers' development of a constructivist view of mathematics learning. *Teaching and Teacher Education*, 8, 187–197.
Simon, M.A. & Schifter, D.E. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics*, 22, 309–331.
Thompson, A., Phillipp, R., Thompson, P. & Boyd, B. (1994). Calculational and conceptual orientations in teaching mathematics. In D. Aichele & A. Coxford (Eds.), *Professional development for teachers of mathematics: 1994 Yearbook NCTM* (79–92). Reston, VA: National Council of Teachers of Mathematics.
Wood, T., Cobb, P. & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, 28, 587–616.

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